J Math Chem (2014) 52:2030–2051
DOI 10.1007/s10910-014-0363-8

ORIGINAL PAPER

# A new efficient technique for solving two-point boundary value problems for integro-differential equations

**Randhir Singh · Gnaneshwar Nelakanti · Jitendra Kumar**

Received: 25 February 2014 / Accepted: 29 April 2014 / Published online: 10 May 2014


**Abstract** In this paper, we propose a new efficient method based on a combination of Adomian decomposition method (ADM) and Green's function for solving second-order boundary value problems (BVPs) for integro-differential equations (IDEs). The proposed method depends on constructing Green's function before establishing the recursive scheme for the solution components. Unlike the ADM or modified ADM , the proposed method avoids solving a sequence of difficult nonlinear equations (transcendental equations) for the unknown parameters. The proposed method provides a direct recursive scheme for obtaining the series solution with easily calculable components. We also provide a sufficient condition that guarantees a unique solution to the second-order BVPs for IDEs. Convergence and error analysis of the proposed method are also discussed. Convergence analysis is reliable enough to estimate the error bound of the series solution. Some numerical examples are included to demonstrate the accuracy, applicability, and generality of the proposed approach. The numerical results reveal that the proposed method is very effective and simple.

Keyword Integro-differential equations · Boundary value problems · Adomian decomposition method · Green's function · Approximations

## 1 Introduction

Fast and accurate numerical solution of boundary value problems (BVPs) for ordinary differential equations is necessary in many branches of applied mathematics, physics and chemistry, e.g., heat and mass transfer within porous catalyst particle [1], oxygen diffusion in cells [2], astrophysics, hydrodynamic and hydromagnetics stability,

R. Singh (✉)· G. Nelakanti · J. Kumar
Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur 721302, India
e-mail: randhir.math@gmail.com; randhir.math@maths.iitkgp.ernet.in

boundary layer theory, the study of stellar interiors, control and optimization theory, and flow networks in biology. In particular, initial and boundary value problems for integro-differential equations (IDEs) arise in chemical engineering, underground water flow and population dynamics, and other field of physics and mathematical chemistry, (for details see, [3–7]). Since it is usually impossible to obtain the closed-form solutions to BVPs of IDEs met in practice, these problems must be solved by various approximate and numerical methods. For details about the existence and uniqueness of solutions for such problems, readers are referred to Agarwal [8].

The aim of this article is to propose an efficient method for solving second-order two-point BVPs for IDEs. The proposed technique is based on a combination of the ADM and Green's function. Consider the following class of second-order two-point BVPs for IDEs as

$$y''(x) = g(x) + \int_a^x K_1(x,t) f_1(t, y(t))dt + \int_a^b K_2(x,t) f_2(t, y(t))dt, \ x \in [a, b], \tag{1.1}$$

subject to the boundary conditions:

$$y(a) = \alpha_1, \quad y(b) = \beta_1, \tag{1.2}$$

where $\alpha_1$ and $\beta_1$ are any finite real constants, $g(x) \in C[a, b]$, and $K_i(x,t) \in C([a, b] \times [a, b]), i = 1, 2$. We assume the following conditions on $f_1(t, y)$ and $f_2(t, y)$:

**(F − 1)** $f_1(t, y), f_2(t, y) \in C\{[a, b] \times \mathbb{R}\}$;
**(F − 2)** $f_1(t, y), f_2(t, y)$ satisfy the Lipschitz condition, i.e, there exists constants $L_1$ and $L_2$ such that

$$|f_1(t, y) - f_1(t, z)| \le L_1|y - z|, \ |f_2(t, y) - f_2(t, z)| \le L_2|y - z|. \tag{1.3}$$

In recent years, a great deal of numerical methods have been applied to solve the particular form of (1.1)–(1.2) in [7–21] and many of the references therein. For example, compact finite difference [14], monotone iterative methods [7], spline collocation method [16], the method of upper and lower solution [17] and Haar wavelets [18] have been studied. Although, these numerical techniques have many advantages, but a huge amount of computational work is involved that combines some root-finding techniques to obtain accurate numerical solution especially for nonlinear problems.

Recently, some newly developed numerical-approximate methods have also been applied to handle some particular cases of the problem (1.1)–(1.2) such as, the Adomian decomposition method (ADM), Laplace ADM (LADM) [12,19], homotopy analysis method (HAM) [20], homotopy perturbation method (HPM) [11] and the variational iteration method [21].

It is well-known that the ADM allows us to solve both nonlinear initial value problems (IVPs) and BVPs without unphysical restrictive assumptions such as linearization, discretization, perturbation and guessing the initial term or a set of basis function. In recent years, many authors [12,19,22–32] have shown interest to study of the ADM for different scientific models. According to the ADM, the problem (1.1) can be written in an operator form as

$$\mathcal{L}y(x) = g(x) + Ny(x), \quad x \in I, \tag{1.4}$$

where $\mathcal{L} = \frac{d^2}{dx^2}$ is a linear second-order differential operator, $g(x)$ is a source function and $Ny(x) = \int_a^x K_1(x,t)f_1(t,y(t))dt + \int_a^b K_2(x,t)f_2(t,y(t))dt$ is a nonlinear operator.

The inverse operator of $\mathcal{L}^{-1}$ is defined as

$$\mathcal{L}^{-1}[\cdot] = \int_a^x \int_a^x [\cdot]dxdx. \tag{1.5}$$

Operating with $\mathcal{L}^{-1}[\cdot]$ on both sides of (1.4) and applying the condition $y(a) = \alpha_1$, we obtain

$$y(x) = \alpha_1 + (x-a)c + \mathcal{L}^{-1}[g(x)] + \mathcal{L}^{-1}[Ny(x)], \tag{1.6}$$

where $c = y'(a)$ is unknown parameter to be determined.

Next the solution $y(x)$, the nonlinear terms $f_1(t,y)$ and $f_2(t,y)$ are decomposed by the finite series as

$$y(x) = \sum_{j=0}^{\infty} y_j(x), \quad f_1(t,y) = \sum_{j=0}^{\infty} A_j \text{ and } f_2(t,y) = \sum_{j=0}^{\infty} B_j, \tag{1.7}$$

where $A_j$ and $B_j$ are Adomian's polynomials which can be obtained by using the formula given in [28] as

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[f\left(t, \sum_{k=0}^{\infty} y_k\lambda^k\right)\right]_{\lambda=0}, \quad n = 0, 1, 2, \ldots. \tag{1.8}$$

Several algorithms have been given to generate the Adomian polynomial rapidly in [33–37].

Substituting the series (1.7) in (1.6), we get

$$\sum_{j=0}^{\infty} y_j(x) = \alpha_1 + (x-a)c + \mathcal{L}^{-1}[g(x)] + \mathcal{L}^{-1}\left[\int_a^x K_1(x,t)\left(\sum_{j=0}^{\infty} A_j\right)dt + \int_a^b K_2(x,t)\left(\sum_{j=0}^{\infty} B_j\right)dt\right]. \tag{1.9}$$

By comparing both sides of (1.9), the ADM is given as

$$\left.\begin{aligned}
&y_0(x,c)=\alpha_1+(x-a)c+\mathcal{L}^{-1}[g(x)],\\
&y_j(x,c)=\mathcal{L}^{-1}\Bigg[\int_a^x K_1(x,t)A_{j-1}dt+\int_a^b K_2(x,t)B_{j-1}dt\Bigg],\ j=1,2\ldots.
\end{aligned}\right\} \tag{1.10}$$

Wazwaz [38] suggested the modified ADM (MADM) as

$$\left.\begin{aligned}
&y_0(x,c)=\alpha_1,\\
&y_1(x,c)=(x-a)c+\mathcal{L}^{-1}[g(x)]+\mathcal{L}^{-1}\Bigg[\int_a^x K_1(x,t)A_0dt+\int_a^b K_2(x,t)B_0dt\Bigg],\\
&y_j(x,c)=\mathcal{L}^{-1}\Bigg[\int_a^x K_1(x,t)A_{j-1}dt+\int_a^b K_2(x,t)B_{j-1}dt\Bigg],\ j=2,3\ldots.
\end{aligned}\right\} \tag{1.11}$$

The recursive schemes (1.10) and (1.11) give the complete determination of the solution components $y_j(x,c)$ of the solution $y(x)$ and the $n$-terms series solution can be obtained as

$$\phi_n(x,c)=\sum_{j=0}^{n} y_j(x,c). \tag{1.12}$$

Note that the series solution $\phi_n(x,c)$ depends on the unknown parameter $c$ which can be obtained approximately by imposing the other boundary condition at $x=b$. This leads to a sequence of nonlinear (or transcendental) equations, i.e., $\phi_n(b,c)=\beta_1, n=1,2,3,\ldots$. For example, consider

$$\left.\begin{aligned}
&y''(x)=-\frac{3}{2}-\frac{1}{(x+1)^2}+\int_0^1 e^{y(t)}dt,\\
&y(0)=0,\quad y(1)=\ln(2).
\end{aligned}\right\} \tag{1.13}$$

According to the modified ADM (1.11), we transform (1.13) into the following recursive scheme as

$$\left.\begin{aligned}
&y_0(x,c)=cx,\\
&y_j(x,c)=-\mathcal{L}^{-1}\left[\tfrac{3}{2}+\tfrac{1}{(x+1)^2}-\int_0^1 A_{j-1}dt\right],\quad j=1,2\ldots
\end{aligned}\right\} \tag{1.14}$$

Using the formula (1.8), the Adomian's polynomials for $e^{y(x)}$ about $y_0=cx$ are given as

$$A_0=e^{cx},\ A_1=y_1(x)ce^{cx},\ A_2=y_2(x)ce^{cx}+\frac{1}{2}y_1^2(x)c^2e^{cx},\ldots \tag{1.15}$$

Using (1.14) and (1.15), the solution components are obtained as

$$\left.\begin{array}{l}
y_0(x,c) = cx,\\
y_1(x,c) = -x - \dfrac{3x^2}{4} - \dfrac{x^2}{2c} + \dfrac{e^c x^2}{2c} + \ln(1+x),\\
y_2(x,c) = -\dfrac{e^c x^2}{c^4} - \dfrac{e^c x^2}{4c^3} + \dfrac{e^c x^2}{c^2} - \dfrac{7e^c x^2}{8c} + \dfrac{e^c x^2\cosh c}{c^4} + \dfrac{e^c x^2\cosh c}{4c^3} - \dfrac{e^c x^2\cosh c}{4c^2}\\
\quad + \dfrac{e^{-c}x^2\mathrm{Ei}(2c)}{2c} - \dfrac{e^{-c}x^2\mathrm{Ei}(2c)}{2c} + \dfrac{e^c x^2\ln 16}{8c} - \dfrac{5e^c x^2\sinh c}{4c^3} + \dfrac{3e^c x^2\sinh c}{4c^2},\\
\vdots
\end{array}\right\} \tag{1.16}$$

Consequently, the $n$-terms series solution can be obtained as $\phi_n(x,c) = \sum_{j=0}^{n} y_j(x,c)$. Now by imposing the other boundary condition at $x = 1$ on $\phi_n(x,c)$, we have a sequence of transcendental equations $\phi_n(1,c) = \ln 2, n = 1, 2, \ldots,$ as follows

$$\left.\begin{array}{l}
\phi_1(1,c) \equiv c - 1 - \dfrac{3}{4} - \dfrac{1}{2c} + \dfrac{e^c}{2c} + \ln(2) = \ln(2),\\
\phi_2(1,c) \equiv c - 1 - \dfrac{3}{4} - \dfrac{1}{2c} + \dfrac{e^c}{2c} + \ln(2) - \dfrac{e^c}{c^4} - \dfrac{e^c}{4c^3} + \dfrac{e^c}{c^2} - \dfrac{7e^c}{8c} + \dfrac{e^c\cosh c}{c^4}\\
\qquad + \dfrac{e^c\cosh c}{4c^3} - \dfrac{e^c\cosh c}{4c^2} + \dfrac{e^{-c}\mathrm{Ei}(2c)}{2c} - \dfrac{e^{-c}\mathrm{Ei}(2c)}{2c} + \dfrac{e^c\ln 16}{8c}\\
\qquad - \dfrac{5e^c\sinh c}{4c^3} + \dfrac{3e^c\sinh c}{4c^2} = \ln(2),\\
\vdots
\end{array}\right\} \tag{1.17}$$

In order to find unknown parameter $c$ from above Eq. (1.17), we need some root finding techniques such as Newton's method which require additional computational work. However, solving a sequence of transcendental Eq. (1.17) for $c$ is a difficult task in general. Moreover, in some cases the unknown parameter $c$ may not be uniquely determined. This may be the main disadvantage of the ADM or the modified ADM for solving BVPs for IDE.

The purpose of this paper is to introduce a modification of the ADM which combines with Green's function technique to overcome the difficulties occurring in the ADM or the MADM for solving second-order BVPs for IDEs of the form (1.1)–(1.2). The proposed method relies on constructing Green's function before establishing the recursive scheme for the solution components. Unlike the ADM or the MADM, the proposed method avoids solving a sequence of nonlinear (or transcendental) Eq. (1.17) for the unknown parameter $c$. We provide the direct recursive scheme for obtaining the approximate solutions in the form of series with easily computable components of the boundary value problem (1.1)–(1.2) without linearization and discretization. Furthermore, we also provide a sufficient condition that guarantees a unique solution to the second-order BVPs for IDEs. Convergence and error analysis of the proposed method are also established. Convergence analysis is reliable enough to estimate the maximum absolute truncated error of the series solution. The reliability and efficiency of the proposed methods are demonstrated by several numerical examples.

## 2 Description of the new technique

In this section we propose an efficient recursive scheme based on a combination of the ADM and Green's function (ADMGF) for solving second-order BVPs for IDEs of the form (1.1)–(1.2). We first consider the following homogeneous boundary value problem as

$$u''(x) = 0, \;\; u(a) = \alpha_1, \; u(b) = \beta_1. \tag{2.1}$$

The exact solution of the problem (2.1) is given by

$$u(x) = \left(\frac{b\alpha_1 - a\beta_1}{b-a}\right) + \left(\frac{\beta_1 - \alpha_1}{b-a}\right)x. \tag{2.2}$$

Now, consider the following second order differential equation with homogeneous boundary conditions as

$$\begin{aligned} y''(x) &= F(x), \; x \in [a, b], \\ y(a) &= 0, \; u(b) = 0, \end{aligned} \tag{2.3}$$

where F $\in C[a, b]$. Integrating (2.3) twice w.r.t. $x$ from $a$ to $x$ and applying the boundary conditions $y(a) = y(b) = 0$, we obtain

$$y(x) = \frac{(a-x)}{b-a}\int_a^b (b-\xi)F(\xi)d\xi + \int_a^x (x-\xi)F(\xi)d\xi.$$

Splitting the first integral into two parts from $a$ to $x$ and $x$ to $b$, we get

$$y(x) = \frac{(a-x)}{b-a}\int_a^x (b-\xi)F(\xi)d\xi + \frac{(a-x)}{b-a}\int_x^b (b-\xi)F(\xi)d\xi + \int_a^x (x-\xi)F(\xi)d\xi.$$

Combining the first and last integrals, we obtain

$$\begin{aligned} y(x) &= \int_a^x \left(\frac{(a-x)(b-\xi)}{b-a} + (x-\xi)\right)F(\xi)d\xi + \frac{1}{b-a}\int_x^b (a-x)(b-\xi)F(\xi)d\xi, \\ &= \frac{1}{b-a}\int_a^x (a-\xi)(b-x)F(\xi)d\xi + \frac{1}{b-a}\int_x^b (a-x)(b-\xi)F(\xi)d\xi, \\ &= \int_a^b G(x,\xi)F(\xi)d\xi, \end{aligned}$$

where the Green's function $G(x,\xi)$ is given by

$$G(x,\xi)=\begin{cases} \dfrac{(a-x)(b-\xi)}{b-a}, & a \le x \le \xi, \\ \dfrac{(a-\xi)(b-x)}{b-a}, & \xi \le x \le b, \end{cases} \tag{2.4}$$

and its maximum value is given by

$$\max_{a\le x,\xi\le b} |G(x,\xi)| = \frac{(b-a)}{4}. \tag{2.5}$$

Using (2.2) and (2.4), we transform boundary value problem for IDEs (1.1)–(1.2) into an equivalent integral equation as

$$y(x) = \left(\frac{b\alpha_1 - a\beta_1}{b-a}\right) + \left(\frac{\beta_1 - \alpha_1}{b-a}\right)x + \int_a^b G(x,\xi)\Bigg[g(\xi) + \int_a^\xi K_1(\xi,t) f_1(t, y(t))dt, \\ + \int_a^b K_2(\xi,t) f_2(t, y(t))dt\Bigg]d\xi, \tag{2.6}$$

Now we apply the ADM to the above Eq. (2.6). Let $y(x)$, $f_1(t,y)$ and $f_2(t,y)$ be represented by the series of components and the Adomian polynomials, respectively as

$$y(x) = \sum_{j=0}^{\infty} y_j(x),\ \ f_1(t,y) = \sum_{j=0}^{\infty} A_j \text{ and } f_2(t,y) = \sum_{j=0}^{\infty} B_j, \tag{2.7}$$

where $A_j$ and $B_j$ are Adomian's polynomials.
Substituting the series (2.7) into (2.6), we obtain

$$\sum_{j=0}^{\infty} y_j(x) = \left(\frac{b\alpha_1 - a\beta_1}{b-a}\right) + \left(\frac{\beta_1 - \alpha_1}{b-a}\right)x + \int_a^b G(x,\xi) \\ \times \left[g(\xi) + \int_a^\xi K_1(\xi,t)\left(\sum_{j=0}^{\infty} A_j\right)dt + \int_a^b K_2(\xi,t)\left(\sum_{j=0}^{\infty} B_j\right)dt\right]d\xi. \tag{2.8}$$

Comparing both sides of (2.8), we obtain the ADMGF for (1.1)–(1.2) as

$$\left.\begin{aligned} y_0(x) &= \left(\frac{b\alpha_1 - a\beta_1}{b-a}\right) + \left(\frac{\beta_1 - \alpha_1}{b-a}\right)x + \int_a^b G(x,\xi) g(\xi) d\xi, \\ y_j(x) &= \int_a^b G(x,\xi)\left[\int_a^\xi K_1(\xi,t) A_{j-1} dt + \int_a^b K_2(\xi,t) B_{j-1} dt\right]d\xi,\ \ j=1,2\ldots \end{aligned}\right\} \tag{2.9}$$

Further by rearranging the terms of $y_0(x)$ and $y_1(x)$ in the above recursive scheme, we have the modified ADMGF for (1.1)-(1.2) as

$$\left.\begin{aligned} y_0(x) &= \left(\frac{b\alpha_1 - a\beta_1}{b-a}\right), \\ y_1(x) &= \left(\frac{\beta_1 - \alpha_1}{b-a}\right)x + \int_a^b G(x,\xi)\left[g(\xi) + \int_a^\xi K_1(\xi,t)A_0 dt + \int_a^b K_2(\xi,t)B_0 dt\right]d\xi, \\ y_j(x) &= \int_a^b G(x,\xi)\left[\int_a^\xi K_1(\xi,t)A_{j-1}dt + \int_a^b K_2(\xi,t)B_{j-1}dt\right]d\xi,\ \ j = 2,3,\ldots, \end{aligned}\right\} \tag{2.10}$$

The recursive schemes (2.9) and (2.10) give the complete determination of the solution components $y_j(x)$ of the solution $y(x)$, and the $n$-terms truncated series solution can be obtained as

$$\psi_n(x) = \sum_{j=0}^{n} y_j(x). \tag{2.11}$$

*Remark 2.1* Unlike the ADM or the MADM, the proposed methods (2.9) and (2.10) avoid unnecessary evaluation of unknown parameters and provide the direct recursive schemes to obtain series solution of the problem (1.1)–(1.2).

*Remark 2.2* It can be noted that the ADMGF (2.9) gives good approximate solution when the problem is linear or nonlinear of the form $y^n$, $yy'$, $y'^n$... while the modified ADMGF (2.10) is useful when the nonlinear function is of the form $e^y$, $\ln y$, $\sin y$, $\cosh y$... etc.

## 3 Convergence analysis

It should be noted that the authors [39,40] have already discussed the convergence of the ADM for differential and integral equations. In this section we discuss the convergence and error analysis of the recursive schemes (2.9) and (2.10) for solving (1.1)–(1.2). Let $\mathbb{X} = C[a,b]$ be the Banach space with the norm

$$\|y\| = \max_{a\le x\le b} |y(x)|,\ \ y \in \mathbb{X}. \tag{3.1}$$

We first rewrite the integral Eq. (2.6) in an operator form as

$$y = \mathcal{N}y, \tag{3.2}$$

where $\mathcal{N} : \mathbb{X} \to \mathbb{X}$ is a nonlinear integral operator given by

$$\begin{aligned} \mathcal{N}y = \left(\frac{b\alpha_1 - a\beta_1}{b-a}\right) + \left(\frac{\beta_1 - \alpha_1}{b-a}\right)x + \int_a^b G(x,\xi)\left[g(\xi) + \int_a^\xi K_1(\xi,t)f_1(t,y(t))dt\right. \\ \left. + \int_a^b K_2(\xi,t)f_2(t,y(t))dt\right]d\xi. \end{aligned} \tag{3.3}$$

In the following theorem we provide the sufficient condition that guarantees a unique solution to the operator Eq. (3.2).

**Theorem 3.1** *Let $\mathbb{X}$ be a Banach space with the norm defined by* (3.1) *and let $\mathcal{N} : \mathbb{X} \to \mathbb{X}$ be the nonlinear integral operator defined by* (3.3) *with the kernels $K_i(\xi, t) \in C([a, b] \times [a, b])$ and $M_i = \max |K_i(\xi, t)|, i = 1, 2$. Assume that the nonlinear functions $f_1(t, y)$ and $f_2(t, y)$ satisfy the Lipschitz condition* (1.3) *with the Lipschitz constants $L_1$ and $L_2$, respectably. If $\delta := \frac{(b-a)^3(L_1M_1+L_2M_2)}{4} < 1$, then the operator Eq.* (3.2) *has a unique solution $y$ in $\mathbb{X}$.*

*Proof* For any $y, y^* \in \mathbb{X}$, consider

$$\|\mathcal{N}y - \mathcal{N}y^*\| = \max_{a\le x\le b}\left|\int_a^b G(x,\xi)\left[\int_a^\xi K_1(\xi,t)[f_1(t,y(t)) - f_1(t,y^*(t))]dt\right.\right.$$
$$\left.\left.+\int_a^b K_2(\xi,t)[f_2(t,y(t)) - f_2(t,y^*(t))]dt\right]d\xi\right|.$$

Now the using Lipschitz conditions of $f_1$ and $f_2$, we obtain

$$\|\mathcal{N}y - \mathcal{N}y^*\| \le \max_{a\le x,\xi\le b}|G(x,\xi)|\left(\int_a^b\left[\max_{a\le\xi,t\le b}|K_1(\xi,t)|\int_a^\xi L_1|y(t)-y^*(t)|dt\right.\right.$$
$$\left.\left.+\max_{a\le\xi,t\le b}|K_2(\xi,t)|\int_a^b L_2|y(t)-y^*(t)|dt\right]d\xi\right).$$

Using the estimate (2.5), it follows

$$\begin{aligned}\|\mathcal{N}y-\mathcal{N}y^*\| &\le \frac{(b-a)}{4}\max_{a\le t\le b}|y(t)-y^*(t)|\left(\int_a^b\left[L_1M_1\int_a^\xi dt+L_2M_2\int_a^b dt\right]d\xi\right)\\ &\le \frac{(b-a)}{4}\|y-y^*\|\left[L_1M_1(b-a)^2+L_2M_2(b-a)^2\right]\\ &= \frac{(b-a)^3\,(L_1M_1+L_2M_2)}{4}\|y-y^*\| = \delta\|y-y^*\|. \end{aligned} \tag{3.4}$$

If $\delta = \frac{(b-a)^3(L_1M_1+L_2M_2)}{4} < 1$ then the mapping $\mathcal{N} : \mathbb{X} \to \mathbb{X}$ is contraction. Hence by the Banach contraction principle, the operator Eq. (3.2) has a unique solution $y$ in $\mathbb{X}$. □

In the following theorem we give the convergence of the series solutions $\psi_n$ obtained from the ADMGF (2.9) or the modified ADMGF (2.10) to the exact solution $y$ of the operator Eq. (3.2).

**Theorem 3.2** *Assume that all the conditions of Theorem 3.1 hold. Let $y_0, y_1, y_2, \ldots,$ be the solution components of the solution $y$ obtained from the ADMGF* (2.9) *or the modified ADMGF* (2.10)*, and let $\psi_n = \sum_{j=0}^n y_j$ be the n-terms series solution defined by* (2.11)*. Then $\psi_n$ converges to the exact solution $y$ of the operator Eq.* (3.2) *whenever $\delta < 1$ and $\|y_1\| < \infty$.*

*Proof* From the ADMGF (2.9) or the modified (2.10) and (2.11), we have

$$\begin{aligned}\psi_n &= y_0 + \sum_{j=1}^{n} y_j, \\ &= \left(\frac{b\alpha_1 - a\beta_1}{b-a}\right) + \left(\frac{\beta_1 - \alpha_1}{b-a}\right)x + \int_a^b G(x,\xi)g(\xi)d\xi \\ &\quad + \sum_{j=1}^{n}\Bigg[\int_a^b G(x,\xi)\Bigg[\int_a^{\xi} K_1(\xi,t)A_{j-1}dt \\ &\quad + \int_a^b K_2(\xi,t)B_{j-1}dt\Bigg]d\xi\Bigg] \\ &= \left(\frac{b\alpha_1 - a\beta_1}{b-a}\right) + \left(\frac{\beta_1 - \alpha_1}{b-a}\right)x + \int_a^b G(x,\xi)\Bigg[g(\xi) + \int_a^{\xi} K_1(\xi,t)\sum_{j=0}^{n-1} A_j dt \\ &\quad + \int_a^b K_2(\xi,t)\sum_{j=0}^{n-1} B_j dt\Bigg]d\xi. \end{aligned} \tag{3.5}$$

Using the relations $\sum_{j=0}^{n} A_j \le f_1(t,\psi_n)$ and $\sum_{j=0}^{n} B_j \le f_2(t,\psi_n)$ in (3.5) as given in ([37] pp. 945), we obtain

$$\begin{aligned}\psi_n \le \left(\frac{b\alpha_1 - a\beta_1}{b-a}\right) + \left(\frac{\beta_1 - \alpha_1}{b-a}\right)x + \int_a^b G(x,\xi)\Bigg[g(\xi) + \int_a^{\xi} K_1(\xi,t)f_1(t,\psi_{n-1})dt \\ + \int_a^b K_2(\xi,t)f_2(t,\psi_{n-1})dt\Bigg]d\xi. \end{aligned} \tag{3.6}$$

Hence for any $n \in \mathbb{N}$ and following the steps of the theorem 3.1, we get

$$\|\psi_{n+1} - \psi_n\| \le \delta\|\psi_n - \psi_{n-1}\|.$$

Thus we have

$$\|\psi_{n+1} - \psi_n\| \le \delta\|\psi_n - \psi_{n-1}\| \le \delta^2\|\psi_{n-1} - \psi_{n-2}\| \le \ldots \le \delta^n\|\psi_1 - \psi_0\|.$$

For all $n, m \in \mathbb{N}$, with $n > m$, consider

$$\begin{aligned}\|\psi_n - \psi_m\| &= \|(\psi_n - \psi_{n-1}) + (\psi_{n-1} - \psi_{n-2}) + \cdots + (\psi_{m+1} - \psi_m)\| \\ &\le \|\psi_n - \psi_{n-1}\| + \|\psi_{n-1} - \psi_{n-2}\| + \cdots + \|\psi_{m+1} - \psi_m\|\end{aligned}$$

$$\leq [\delta^{n-1} + \delta^{n-2} + \cdots + \delta^m]\|\psi_1 - \psi_0\|$$
$$= \delta^m[1 + \delta + \delta^2 + \cdots + \delta^{n-m-1}]\|\psi_1 - \psi_0\|$$
$$= \delta^m \left(\frac{1 - \delta^{n-m}}{1 - \delta}\right) \|y_1\|.$$

Since $\delta < 1$ so, $(1 - \delta^{n-m}) < 1$ and $\|y_1\| < \infty$, it follows that

$$\|\psi_n - \psi_m\| \leq \frac{\delta^m}{1 - \delta}\|y_1\|, \tag{3.7}$$

which converges to zero, i.e., $\|\psi_n - \psi_m\| \to 0$, as $m \to \infty$. This implies that there exits a $\psi$ such that $\lim_{n\to\infty} \psi_n = \psi$. Since, we have $y = \sum_{j=0}^{\infty} y_j = \lim_{n\to\infty} \psi_n$, that is, $y = \psi$ which is the exact solution of (3.2). □

In the following theorem we provide the error bound of the series solution (2.11) obtained through the ADMGF (2.9).

**Theorem 3.3** *Let $y$ be the exact solution of the operator Eq.* (3.2) *and let $\psi_m$ be a sequence of approximate series solutions* (2.11) *obtained from the ADMGF* (2.9). *Then there holds*

$$\|y - \psi_m\| \leq \frac{\delta^m}{(1 - \delta)} C_1,$$

*where* $C_1 = \dfrac{(b-a)^3}{4} \max_{a\leq t\leq b} (M_1|f_1(t, y_0)| + M_2|f_2(t, y_0)|)$.

*Proof* Since $\lim_{n\to\infty} \psi_n = y$, fixing $m$ and letting $n \to \infty$ in (3.7) with $n \geq m$, we obtain

$$\|y - \psi_m\| \leq \frac{\delta^m}{1 - \delta}\|y_1\|. \tag{3.8}$$

Since $y_1(x) = \int_a^b G(x,\xi)\left(\int_a^\xi K_1(\xi,t)A_0 dt + \int_a^b K_2(\xi,t)B_0 dt\right)d\xi$, $A_0 = f_1(t, y_0)$, $B_0 = f_2(t, y_0)$, we have

$$\|y_1\| \leq \max_{a\leq x\leq b} \int_a^b |G(x,\xi)|\left(\int_a^\xi |K_1(\xi,t)||f_1(t,y_0)|dt + \int_a^b |K_2(\xi,t)||f_2(t,y_0)|dt\right)d\xi$$
$$\leq \frac{(b-a)^3}{4} \max_{a\leq t\leq b} (M_1|f_1(t, y_0)| + M_2|f_2(t, y_0)|). \tag{3.9}$$

Combining the estimates (3.8) and (3.9), we get desired result of the theorem. □

Similarly, we can obtain the error bound of the series solution (2.11) obtained through the modified ADMGF (2.10).

**Theorem 3.4** *Let $y$ be the exact solution of the operator Eq.* (3.2) *and let $\psi_m$ be a sequence of approximate series solutions* (2.11) *obtained from the modified ADMGF* (2.10). *Then there holds*

$$\|y-\psi_m\| \leq \frac{\delta^m}{(1-\delta)}C_2,$$

*where* $C_2 = \left(\frac{\beta_1-\alpha_1}{b-a}\right)b + \frac{(b-a)^3}{4}\max_{a\leq t\leq b}(M_1|f_1(t,y_0)| + M_2|f_2(t,y_0)|)$.

## 4 Numerical results and discussion

In this section we consider three examples to demonstrate the accuracy and efficiency of the modified ADMGF (2.10). All symbolic and numerical computations are performed by using 'Mathematica' 8.0 software package. Numerical results obtained by the proposed method are compared with the exact and known results.

*Example 4.1* Consider the following nonlinear second-order BVPs for IDE

$$\left.\begin{array}{l} y''(x) = g(x) + \int\limits_0^x e^{-t}y^2(t)dt, \quad x\in[0,1], \\ y(0)=1, \quad y(1)=e, \end{array}\right\} \tag{4.1}$$

where $g(x)=1$ and the exact solution is $y(x)=e^x$.

According to the modified ADMGF (2.10) with $\alpha_1=1, \beta_1=e, K_1(x,t)=e^{-t}, f_1(t,y)=y^2(t)$, $K_2(x,t)=0$, and $f_2(t,y)=0$, we transform the problem (4.1) into the following recursive scheme as

$$\left.\begin{array}{l} y_0(x)=1, \\ y_1(x) = (e-1)x + \int\limits_0^1 G(x,\xi)\left[g(\xi) + \int\limits_0^\xi K_1(\xi,t)A_0dt\right]d\xi, \\ y_j(x) = \int\limits_0^1 G(x,\xi)\left[\int\limits_0^\xi K_1(\xi,t)A_{j-1}dt\right]d\xi, \quad j=2,3,\ldots \end{array}\right\} \tag{4.2}$$

where the Green's function $G(x,\xi)$ is given by

$$G(x,\xi) = \begin{cases} -x(1-\xi), & 0\leq x\leq \xi, \\ -\xi(1-x), & \xi\leq x\leq 1. \end{cases} \tag{4.3}$$

For fast computer generation, we use the Duan's efficient algorithm [41] for obtaining the Adomian's polynomial for $f_1 = y^2$ given as

$$A_0 = y_0^2,\ A_1 = 2y_0y_1,\ A_2 = y_1^2 + 2y_0y_2, \ldots \tag{4.4}$$

Using (4.2) and (4.4), we obtain the solution components as

$$\begin{aligned}
y_0(x) &= 1,\\
y_1(x) &= 1 - e^{-x} + 0.0861613x + x^2,\\
y_2(x) &= 26.266967{+}0.2500e^{-2x}{-}26.516967e^{-x} - 13.918209x - 12.172322e^{-x}x\\
&\quad + 2.586161x^2 - 2e^{-x}x^2 + 4.440892 \times 10^{-16}x^3,\\
&\ \ \vdots
\end{aligned}$$

In order to show the accuracy and efficiency of the proposed method, we define absolute error function as

$$e_n(x) = |\psi_n(x) - y(x)|,\ n = 1, 2, \ldots$$

where $y(x)$ is the exact solution and $\psi_n(x)$ is $n$-terms series solution obtained from the ADMGF (2.9) or the modified ADMGF (2.10).

Table 1 shows the comparison between maximum absolute errors $|\psi_n - y|$ obtained by the proposed modified ADMGF (2.10) and $|\phi_n - y|$ obtained by the MADM (1.11). It can observed that the our method (2.10) provides not only better numerical results but also avoids solving a sequence of transcendental equations for unknown constant. Moreover, we have also plotted the absolute error functions $e_n(x)$ for $n = 1(1)6$ in Figs. 1 and 2. We also plot the exact $y(x)$ and the approximate solutions $\psi_1$, $\psi_2$ in Fig. 3.

**Table 1** Comparison of the numerical results of Example 4.1

| | Proposed | Method | | MADM | | |
|---|---|---|---|---|---|---|
| $x$ | $\lvert\psi_2 - y\rvert$ | $\lvert\psi_4 - y\rvert$ | $\lvert\psi_6 - y\rvert$ | $\lvert\phi_2 - y\rvert$ | $\lvert\phi_4 - y\rvert$ | $\lvert\phi_6 - y\rvert$ |
| 0.0 | 0.0000E−00 | 0.0000E−00 | 1.2732E−00 | 0.0000E−00 | 0.0000E−00 | 4.2732E−11 |
| 0.1 | 1.2602E−03 | 1.2909E−05 | 2.7033E−07 | 1.2902E−02 | 2.2909E−03 | 3.7033E−04 |
| 0.2 | 2.5246E−03 | 2.6007E−05 | 5.3162E−07 | 2.5646E−02 | 4.5007E−03 | 4.3162E−04 |
| 0.3 | 3.7900E−03 | 3.9497E−05 | 7.6330E−07 | 3.7900E−02 | 2.9497E−03 | 6.6330E−04 |
| 0.4 | 5.0235E−03 | 5.3198E−05 | 9.3734E−07 | 6.0235E−02 | 6.7191E−03 | 9.4734E−04 |
| 0.5 | 6.1392E−03 | 6.6136E−05 | 1.0246E−06 | 7.1392E−02 | 6.6132E−03 | 2.0246E−04 |
| 0.6 | 6.9728E−03 | 7.6176E−05 | 1.0033E−06 | 8.9728E−02 | 8.6173E−03 | 2.4033E−04 |
| 0.7 | 7.2566E−03 | 7.9781E−05 | 8.6709E−07 | 5.2566E−02 | 7.9784E−03 | 9.6729E−04 |
| 0.8 | 6.5922E−03 | 7.2046E−05 | 6.3067E−07 | 5.6922E−02 | 7.2046E−03 | 7.3167E−04 |
| 0.9 | 4.4222E−03 | 4.7215E−05 | 3.2839E−07 | 4.5222E−02 | 6.7214E−03 | 4.2869E−04 |
| 1.0 | 0.0000E−00 | 0.0000E−00 | 0.0000E−00 | 3.1383E−15 | 8.6841E−17 | 6.6067E−14 |

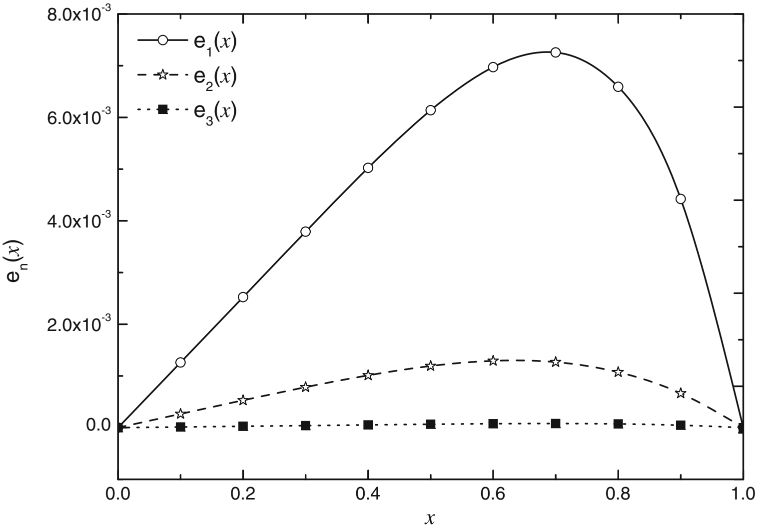


**Fig. 1** Absolute error functions $e_n(x)$ for $n = 1, 2, 3$ of Example 4.1

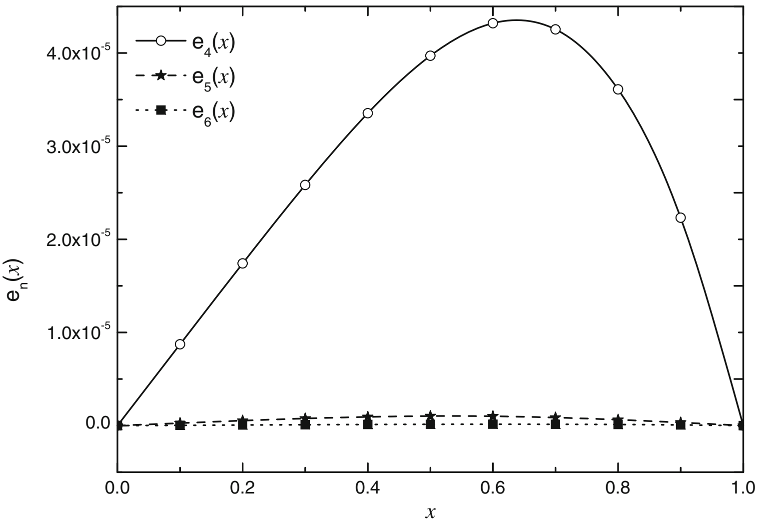


**Fig. 2** Absolute error functions $e_n(x)$ for $n = 4, 5, 6$ of Example 4.1

*Example 4.2* Consider the following nonlinear second-order BVPs for IDE

$$\left.\begin{aligned} & y''(x) = g(x) + \int_0^x (x-t)e^{y(t)}dt, \quad x \in I = [0,1], \\ & y(0) = \ln(4), \quad y(1) = \ln(5), \end{aligned}\right\} \tag{4.5}$$

where $g(x) = -2x^2 - \frac{x^3}{6} - \frac{1}{(4+x)^2}$.

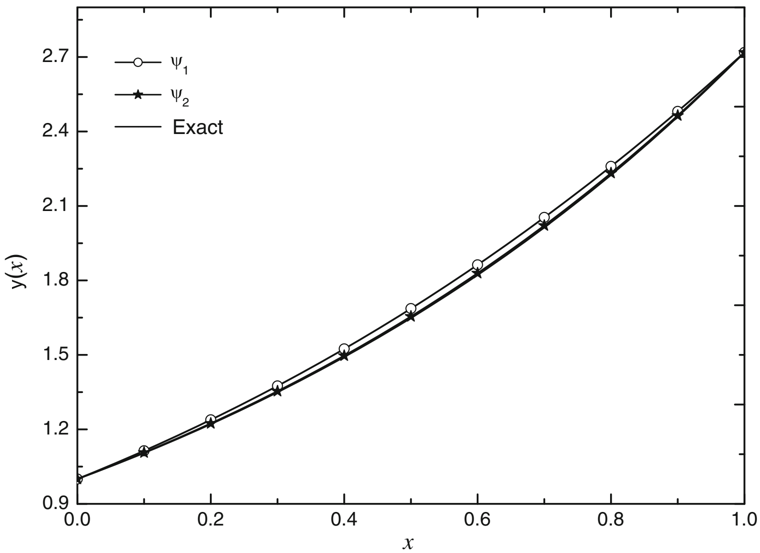


**Fig. 3** Comparison of the exact $y(x)$ and the approximations $\psi_1$, $\psi_2$ of Example 4.1

According to the modified ADMGF (2.10) with $\alpha_1 = \ln(4)$, $\beta_1 = \ln(5)$, $K_1(x,t) = (x-t)$, $f_1(t, y) = e^{y(t)}$, $K_2(x,t) = 0$, and $f_2(t, y) = 0$, the problem (4.5) is transformed into the following recursive scheme as

$$\left.\begin{aligned} &y_0(x) = \ln(4),\\ &y_1(x) = [\ln(5) - \ln(4)]x + \int_0^1 G(x,\xi)\left[g(\xi) + \int_0^\xi K_1(\xi,t)A_0 dt\right]d\xi,\\ &y_j(x) = \int_0^1 G(x,\xi)\left[\int_0^\xi K_1(\xi,t)A_{j-1} dt\right]d\xi, \quad j = 2, 3, \ldots \end{aligned}\right\} \tag{4.6}$$

where the Green's function $G(x, \xi)$ is given by (4.3).

Similarly, using the Duan's efficient algorithm [41], the Adomian's polynomials for $f_1 = e^y$ are calculated as

$$A_0 = e^{y_0},\ A_1 = e^{y_0}y_1,\ A_2 = \frac{1}{2}e^{y_0}y_1^2 + e^{y_0}y_2, \ldots, \tag{4.7}$$

Using (4.6) and (4.7), the solution components are computed as

$$\begin{aligned} &y_0(x) = \ln(4),\\ &y_1(x) = -1.469627x + 0.166666x^2 - 0.083333x^3 + (x-1)\ln\left(\frac{4}{4+x}\right)\\ &\qquad + x\ln(4+x), \end{aligned}$$

**Table 2** Comparison of the numerical results of Example 4.2

| | Proposed | Method | | MADM | | |
|---|---|---|---|---|---|---|
| $x$ | $\|\psi_2 - y\|$ | $\|\psi_4 - y\|$ | $\|\psi_6 - y\|$ | $\|\phi_2 - y\|$ | $\|\phi_4 - y\|$ | $\|\phi_6 - y\|$ |
| 0.0 | 0.0000E−00 | 0.0000E−00 | 0.0000E−00 | 0.0000E−00 | 0.0000E−00 | 0.0000E−00 |
| 0.1 | 6.7283E−04 | 2.2970E−06 | 2.7033E−07 | 5.7282E−02 | 3.2970E−04 | 2.9034E−05 |
| 0.2 | 1.0230E−03 | 2.5066E−06 | 5.3162E−07 | 3.0231E−02 | 3.5066E−04 | 5.4163E−05 |
| 0.3 | 1.1155E−03 | 1.2202E−06 | 7.6330E−07 | 2.1153E−02 | 2.2202E−04 | 7.7331E−05 |
| 0.4 | 1.0153E−03 | 9.7093E−07 | 9.3734E−07 | 2.0151E−02 | 8.7093E−04 | 9.4733E−05 |
| 0.5 | 7.8720E−04 | 3.4754E−06 | 1.0246E−06 | 6.8722E−02 | 5.4754E−04 | 1.8245E−05 |
| 0.6 | 4.9609E−04 | 5.7018E−06 | 1.0033E−06 | 3.9603E−02 | 4.7018E−04 | 1.7034E−05 |
| 0.7 | 2.0690E−04 | 7.0589E−06 | 8.6709E−07 | 4.0692E−02 | 6.0589E−04 | 8.6708E−05 |
| 0.8 | 1.5445E−05 | 6.9552E−06 | 6.3067E−07 | 2.5443E−02 | 5.9552E−04 | 6.6065E−05 |
| 0.9 | 1.0605E−05 | 4.7993E−06 | 3.2839E−07 | 2.0603E−02 | 3.7993E−04 | 3.5838E−05 |
| 1.0 | 0.0000E−00 | 0.0000E−00 | 0.0000E−00 | 3.2203E−16 | 3.2204E−16 | 1.1756E−11 |

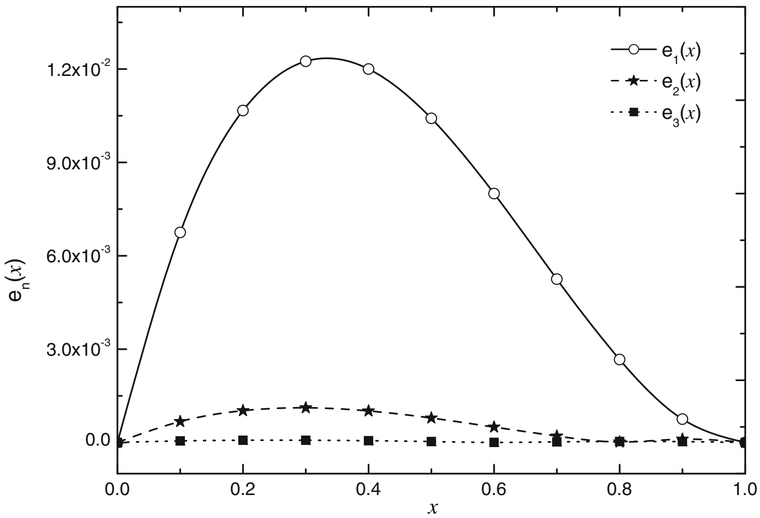


**Fig. 4** Absolute error functions $e_n(x)$ for $n = 1, 2, 3$ of Example 4.2

$$y_2(x) = 0.0747756x - 0.147291x^2 + 0.0725155x^3,$$

$$\vdots$$

Similarly, Table 2 shows the comparison of maximum absolute error $|\psi_n - y|, n = 2, 4, 6$ obtained by the modified ADMGF (2.10) and $|\phi_n - y|, n = 2, 4, 6$ obtained by the MADM (1.11). Once again, it has been shown that the proposed method gives better numerical results compared to the MADM (1.11). Also note that the modified ADMGF avoids extra calculations for unknown constants. Furthermore, we plot error functions $e_n(x)$ for $n = 1(1)6$ in Figs. 4 and 5. Figure 6 shows the plot of the exact solution $y(x)$ and the approximations $\psi_n, n = 1, 2$.

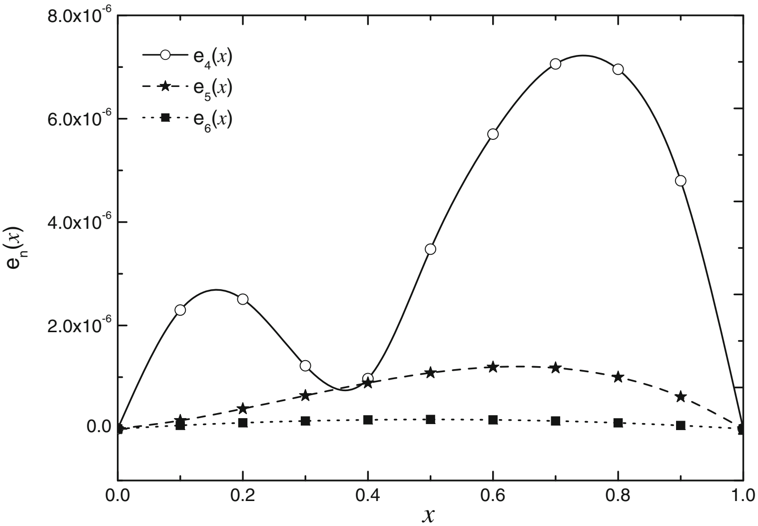


**Fig. 5** Absolute error functions $e_n(x)$ for $n = 4, 5, 6$ of Example 4.2

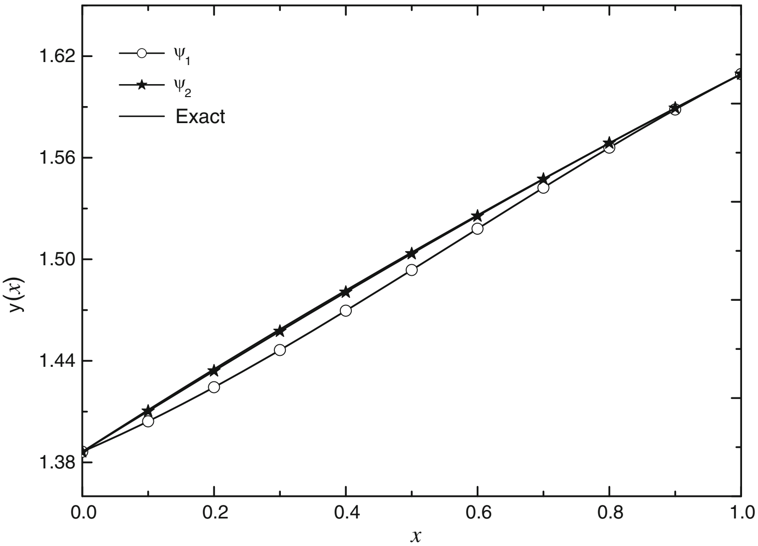


**Fig. 6** Comparison of the exact $y(x)$ and the approximations $\psi_1$, $\psi_2$ of Example 4.2

*Example 4.3* Consider the following nonlinear second-order BVPs for IDE [20]

$$\left.\begin{aligned} &y''(x) = g(x) + \int_0^1 (x-t)e^{y(t)}dt, \quad x \in I = [0, 1], \\ &y(0) = 0, \quad y(1) = \ln(2), \end{aligned}\right\} \tag{4.8}$$

where $g(x) = -\frac{3x}{2} - \frac{1}{(x+1)^2} + \frac{5}{6}$.

**Table 3** Comparison of the numerical results of Example 4.3

| | Proposed | Method | | MADM | | |
|---|---|---|---|---|---|---|
| $x$ | $\lvert\psi_2 - y\rvert$ | $\lvert\psi_4 - y\rvert$ | $\lvert\psi_6 - y\rvert$ | $\lvert\phi_2 - y\rvert$ | $\lvert\phi_4 - y\rvert$ | $\lvert\phi_6 - y\rvert$ |
| 0.0 | 0.0000E−00 | 0.0000E−00 | 0.0000E−00 | 0.0000E−00 | 0.0000E−00 | 0.0000E−00 |
| 0.1 | 1.8842E−03 | 6.3121E−05 | 1.4764E−07 | 1.8842E−02 | 6.3122E−03 | 1.4764E−05 |
| 0.2 | 3.0280E−03 | 1.0072E−05 | 2.2706E−07 | 2.0280E−02 | 4.0071E−03 | 3.2706E−05 |
| 0.3 | 3.5520E−03 | 1.1712E−05 | 2.5154E−07 | 7.5521E−02 | 2.1714E−03 | 5.5154E−05 |
| 0.4 | 3.5769E−03 | 1.1661E−05 | 2.3435E−07 | 6.5768E−02 | 2.1662E−03 | 5.3435E−05 |
| 0.5 | 3.2232E−03 | 1.0352E−05 | 1.8879E−07 | 5.2231E−02 | 2.0351E−03 | 6.8879E−05 |
| 0.6 | 2.6117E−03 | 8.2142E−05 | 1.2812E−07 | 4.6116E−02 | 7.2141E−03 | 7.2812E−05 |
| 0.7 | 1.8629E−03 | 5.6793E−05 | 6.5627E−07 | 4.8628E−02 | 7.6792E−03 | 9.5627E−05 |
| 0.8 | 1.0976E−03 | 3.1780E−05 | 1.4589E−07 | 4.0975E−02 | 3.1781E−03 | 8.4589E−05 |
| 0.9 | 4.3647E−04 | 1.1413E−05 | 1.1712E−07 | 3.3646E−02 | 2.1413E−03 | 7.1712E−05 |
| 1.0 | 0.0000E−00 | 0.0000E−00 | 0.0000E−00 | 2.1102E−16 | 2.1102E−16 | 3.3306E−16 |

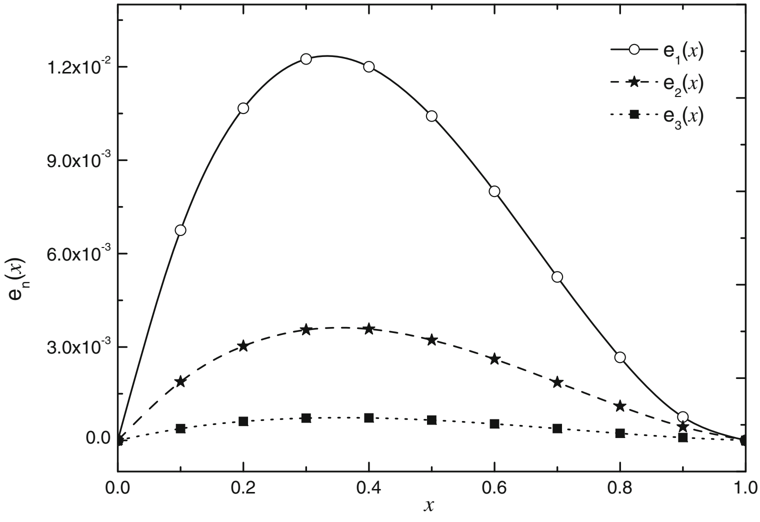


**Fig. 7** Absolute error functions $e_n(x)$ for $n = 1, 2, 3$ of Example 4.3

According to the modified ADMGF (2.10) with $\alpha_1 = 0$, $\beta_1 = \ln(2)$, $K_1(x,t) = 0$, $f_1(t,y) = 0$, $f_2(t,y) = e^{y(t)}$ and $K_2(x,t) = (x-t)$, we transform the problem (4.8) into the following recursive scheme as

$$\left.\begin{aligned} &y_0(x) = 0,\\ &y_1(x) = (\ln(2) - 0)x + \int_0^1 G(x,\xi)\left[g(\xi) + \int_0^1 K_2(\xi,t)A_0 dt\right]d\xi,\\ &y_j(x) = \int_0^1 G(x,\xi)\left[\int_0^1 K_2(\xi,t)A_{j-1}dt\right]d\xi, \quad j = 2,3,\ldots \end{aligned}\right\} \tag{4.9}$$

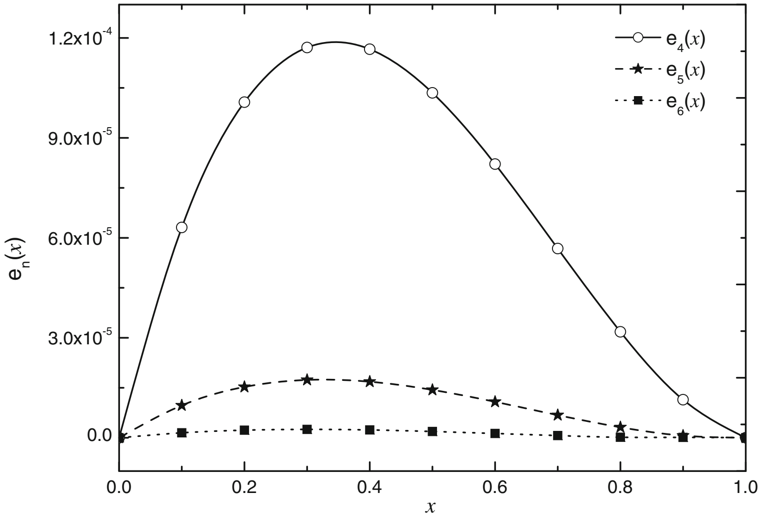


**Fig. 8** Absolute error functions $e_n(x)$ for $n = 4, 5, 6$ of Example 4.3

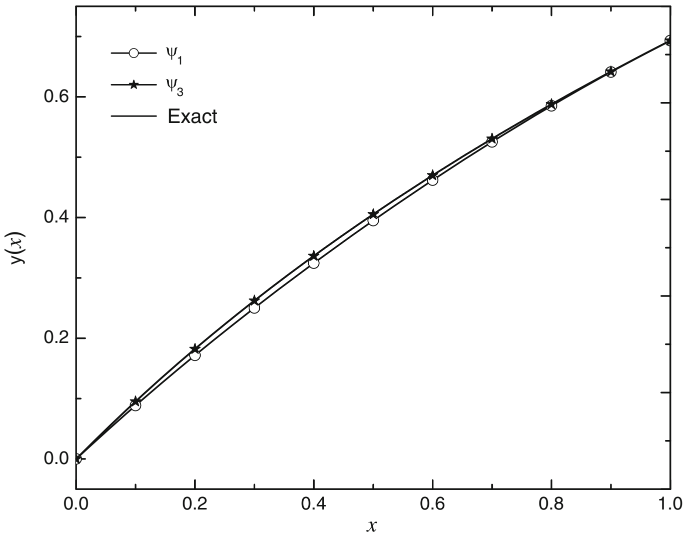


**Fig. 9** Comparison of the exact $y(x)$ and the approximations $\psi_1$, $\psi_3$ of Example 4.3

where the Green's function $G(x, \xi)$ by (4.3) and the Adomian's polynomials $A_j$ are given by (4.7).

Using (4.9) and (4.7), we obtain the solution components as

$$
\begin{aligned}
&y_0(x) = 0,\\
&y_1(x) = -0.083333x + 0.166666x^2 - 0.083333x^3 + \ln(1 + x),
\end{aligned}
$$

$$y_2(x) = 0.0603861x - 0.123611x^2 + 0.063225x^3,$$
$$\vdots$$

Table 3 shows the comparison of maximum absolute error $|\psi_n - y|, n = 2,\ 4,\ 6$ obtained by the modified ADMGF (2.10) and $|\phi_n - y|, n = 2,\ 4,\ 6$ obtained by the MADM (1.11). Like previous examples, the accuracy of the proposed method is tested by plotting the absolute error functions $e_n(x)$ for $n = 1(1)6$ in Figs. 7 and 8. We have also plotted the exact solution $y(x)$ and approximations $\psi_1$ and $\psi_3$ in Fig. 9.

## 5 Conclusion

We have presented a new effective method based on a combination of the Adomian decomposition method and Green's function for solving nonlinear second-order IDEs approximately. It depends on constructing Green's function before establishing the recursive scheme for the solution components of the solution. Unlike the ADM or the MADM, the proposed method (ADMGF or the modified ADMGF) avoids unnecessary evaluation of unknown parameters and provides much better numerical results. The accuracy and efficiency of the proposed method has been tested by solving three numerical examples of second-order BVPs for IDEs. Unlike the finite difference, the cubic spline methods, and any other discretization methods, the proposed method does not require any linearization or discretization of variables. In addition, we have provided a sufficient condition that guarantees a unique solution to the second-order BVPs for IDEs. Convergence and error analysis of the proposed method have also been discussed. Convergence analysis is reliable enough to estimate the error bound of the series solution.

**Acknowledgments** The authors would like to thank the Editor-in-Chief, and the anonymous referees for their useful comments and suggestions that led to improvement of the presentation and content of this paper. The authors thankfully acknowledge the financial assistance provided by Council of Scientific and Industrial Research (CSIR), New Delhi, India.

## References

1. R.C. Rach, J.S. Duan, A.M. Wazwaz, Solving coupled Lane–Emden boundary value problems in catalytic diffusion reactions by the Adomian decomposition method. J. Math. Chem. **52**(1), 255–267 (2014)
2. P.M. Lima, L. Morgado, Numerical modeling of oxygen diffusion in cells with Michaelis-Menten uptake kinetics. J. Math. Chem. **48**(1), 145–158 (2010)
3. K. Maleknejad, I.N. Khalilsaraye, M. Alizadeh, On the solution of the integro-differential equation with an integral boundary condition. Numer. Algorithm. **65**(2), 355–374 (2014)
4. H. Yao, New algorithm for the numerical solution of the integro-differential equation with an integral boundary condition. J. Math. Chem. **47**(3), 1054–1067 (2010)
5. F. Bloom, Asymptotic bounds for solutions to a system of damped integro-differential equations of electromagnetic theory. J. Math. Anal. Appl. **73**(2), 524–542 (1980)
6. A. Jerri, *Introduction to integral equations with applications* (Wiley, Hoboken, 1999)
7. Q.M. Al-Mdallal, Monotone iterative sequences for nonlinear integro-differential equations of second order. Nonlinear Anal.: Real World Appl. **12**(6), 3665–3673 (2011)
8. R.P. Agarwal, Boundary value problems for higher order integro-differential equations. Nonlinear Anal.: Theory Methods Appl. **7**(3), 259–270 (1983)
9. G. Pittaluga, L. Sacripante, An algorithm for solving fredholm integro-differential equations. Numer. Algorithm. **50**(2), 115–126 (2009)

10. I. Cravero, G. Pittaluga, L. Sacripante, An algorithm for solving linear volterra integro-differential equations. Numer. Algorithm. **60**(1), 101–114 (2012)
11. J. Saberi-Nadjafi, A. Ghorbani, He's homotopy perturbation method: an effective tool for solving nonlinear integral and integro-differential equations. Comput. Math. Appl. **58**(11), 2379–2390 (2009)
12. A.M. Wazwaz, A comparison study between the modified decomposition method and the traditional methods for solving nonlinear integral equations. Appl. Math. Comput. **181**(2), 1703–1712 (2006)
13. A. Saadatmandi, M. Dehghan, Numerical solution of the higher-order linear fredholm integro-differential-difference equation with variable coefficients. Comput. Math. Appl. **59**(8), 2996–3004 (2010)
14. J. Zhao, R.M. Corless, Compact finite difference method for integro-differential equations. Appl. Math. Comput. **177**(1), 271–288 (2006)
15. S. Nas, S. Yalcinbas, M. Sezer, A taylor polynomial approach for solving high-order linear fredholm integro-differential equations. Int. J. Math. Educ. **31**(2), 213–225 (2000)
16. H. Brunner, On the numerical solution of nonlinear volterra integro-differential equations. BIT Numer. Math. **13**(4), 381–390 (1973)
17. E. Liz, J.J. Nieto, Boundary value problems for second order integro-differential equations of fredholm type. J. Comput. Appl. Math. **72**(2), 215–225 (1996)
18. Ü. Lepik, Haar wavelet method for nonlinear integro-differential equations. Appl. Math. Comput. **176**(1), 324–333 (2006)
19. A.M. Wazwaz, The combined Laplace transform–Adomian decomposition method for handling nonlinear volterra integro-differential equations. Appl. Math. Comput. **216**(4), 1304–1309 (2010)
20. A. El-Ajou, O. Abu Arqub, S. Momani, Momani, homotopy analysis method for second-order boundary value problems of integro-differential equations. Discret. Dyn. Nat. Soc. **2012**, 1–18 (2012)
21. S.Q. Wang, J.H. He, Variational iteration method for solving integro-differential equations. Phys. Lett. A **367**(3), 188–191 (2007)
22. M. El-Gamel, A comparison between the Sinc–Galerkin and the modified decomposition methods for solving two-point boundary-value problems. J. Comput. Phys. **223**(1), 369–383 (2007)
23. E. Deeba, S. Khuri, S. Xie, An algorithm for solving boundary value problems. J. Comput. Phys. **159**(2), 125–138 (2000)
24. J.S. Duan, R.C. Rach, A new modification of the Adomian decomposition method for solving boundary value problems for higher order nonlinear differential equations. Appl. Math. Comput. **218**(8), 4090–4118 (2011)
25. J.S. Duan, R.C. Rach, A.-M. Wazwaz, T. Chaolu, Z. Wang, A new modified Adomian decomposition method and its multistage form for solving nonlinear boundary value problems with robin boundary conditions. Appl. Math. Model. **37**(2021), 8687–8708 (2013)
26. S. Khuri, A. Sayfy, A novel approach for the solution of a class of singular boundary value problems arising in physiology. Math. Comput. Model. **52**(3), 626–636 (2010)
27. A.M. Wazwaz, A reliable study for extensions of the Bratu problem with boundary conditions. Math. Methods Appl. Sci. **35**(7), 845–856 (2012)
28. G. Adomian, R.C. Rach, Inversion of nonlinear stochastic operators. J. Math. Anal. Appl. **91**(1), 39–46 (1983)
29. G. Adomian, *Solving frontier problems of physics: the decomposition method* (Kluwer Academic Publishers, Berlin, 1994)
30. A.M. Wazwaz, Approximate solutions to boundary value problems of higher order by the modified decomposition method. Comput. Math. Appl. **40**(6–7), 679–691 (2000)
31. M. Benabidallah, Y. Cherruault, Application of the Adomian method for solving a class of boundary problems. Kybernetes **33**(1), 118–132 (2004)
32. R. Singh, J. Kumar, G. Nelakanti, New approach for solving a class of doubly singular two-point boundary value problems using Adomian decomposition method. Adv. Numer. Anal. **2012**, 1–22 (2012)
33. R.C. Rach, A convenient computational form for the Adomian polynomials. J. Math. Anal. Appl. **102**(2), 415–419 (1984)
34. J.S. Duan, Recurrence triangle for Adomian polynomials. Appl. Math. Comput. **216**(4), 1235–1241 (2010)
35. J.S. Duan, An efficient algorithm for the multivariable Adomian polynomials. Appl. Math. Comput. **217**(6), 2456–2467 (2010)
36. A.M. Wazwaz, A new algorithm for calculating Adomian polynomials for nonlinear operators. Appl. Math. Comput. **111**(1), 33–51 (2000)

37. R.C. Rach, A new definition of the Adomian polynomials. Kybernetes **37**(7), 910–955 (2008)
38. A.M. Wazwaz, A reliable modification of Adomian decomposition method. Appl. Math. Comput. **102**(1), 77–86 (1999)
39. A. Abdelrazec, D. Pelinovsky, Convergence of the Adomian decomposition method for initial-value problems. Numer. Methods Partial Differ. Equ. **27**(4), 749–766 (2011)
40. M. Hosseini, H. Nasabzadeh, On the convergence of Adomian decomposition method. Appl. Math. Comput. **182**(1), 536–543 (2006)
41. J.S. Duan, Convenient analytic recurrence algorithms for the Adomian polynomials. Appl. Math. Comput. **217**(13), 6337–6348 (2011)